The energy levels of lattice gauge theory in a small twisted box

David Daniel [1]
Physics Department, University of Edinburgh, Edinburgh EH9 3JZ, Scotland, UK

Antonio González-Arroyo
Departamento de Física Teórica, Universidad Autónoma de Madrid, E-28049 Madrid, Spain

and

Chris P. Korthals Altes [2]
Department of Physics, Boston University, 590 Commonwealth Avenue, Boston MA 02215, USA

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We compute the low-lying energy levels in SU(2) lattice gauge theory with twisted boundary conditions. Lattice artifacts are sizable for the negative parity sector. Convergence to the continuum limit is slowed down due to logarithmic corrections. On the other hand finite size effects are less than in the case of periodic boundary conditions, as borne out by Monte Carlo simulations.

1. Introduction

The last few years have seen a steady accumulation of Monte Carlo data and analytic work in pure (quarkless) QCD and its bound states the glueballs. The strategy is to measure masses as a function of the ratio z of the size L of the finite volume (in which simulations are necessarily performed) to the Compton wavelength of the glueball in question (for a review see ref. [1]). For small enough values of z the results can be compared with perturbative calculations, a simple consequence of asymptotic freedom. The physical mass is obtained for large z where the approach to the infinite volume mass is known to be exponentially fast [2] (for periodic boundary conditions). One obvious question is: what z is large enough? Presumably this is the size at which string formation sets in. By monitoring time correlations between space-like Polyakov loops one can see the advent of such a regime. This way of proceeding has a two-fold aim:

(1) To have control over finite size effects.

(2) More ambitiously, to get from the intermediate regime (between the perturbative and the string domains) an idea of how confinement works.

Some time ago, two of us [3] proposed a method which may at least help with point (1). We introduced boundary conditions that respect the fast approach to the large volume limit, but decrease the finite size effects in the perturbative domain. These are the *twisted boundary conditions* of 't Hooft [4], which are periodic modulo a gauge transformation. We studied pure SU(2) gauge theory with a twist that preserved cubic invariance, therefore making comparison with the usual periodic boundary conditions straightforward. The low-lying energy levels were computed in the continuum limit [5].

In this note these results are extended to a finite lattice spacing. The lay-out of this paper is as follows. In section 2 the calculation of the spectrum is explained. Section 3 deals with the results, and in section 4 a check is made on these results by means of

[1] Address from September 1990: Theoretical Division, T8 (MS B-285), Los Alamos National Laboratory, Los Alamos, NM 87545, USA.

[2] On leave from Centre de Physique Théorique au CNRS, Case 907, Luminy, F-13288 Marseille Cedex 9, France.

the Ward identities. A short comment on the effects of fermion loops is contained in section 5. Finally in section 6 the conclusions are presented and comparison with Monte Carlo results is made.

2. The twisted box

Twisted boundary conditions are boundary conditions which are periodic modulo a gauge transformation. For consistency we require the gauge field to be single valued. Consequently, in an $SU(N)$ gauge theory, permissible twists can be characterized by a matrix, $n_{\mu\nu}$, whose elements are integers modulo N. This measures the commutativity of the gauge transformations which define the boundary conditions [4]. For twists in space–space directions we define 't Hooft's abelian magnetic flux vector $\boldsymbol{m}$ by

$$m_i = \epsilon_{ijk} n_{jk} \,. \tag{1}$$

Another useful concept in discussing gauge theory in a finite volume is the abelian electric flux $\boldsymbol{e}$, which, like $\boldsymbol{m}$, is integral modulo N. The physical Hilbert space decomposes into sectors of given $\boldsymbol{e}$ which are representations of the group of "singular" gauge transformations [periodic modulo the centre group, $Z(N)$], in much the same way as the θ parameter distinguishes sectors related by "large" gauge transformations.

In a small twisted box the perturbative structure of the electric sectors is quite different from the purely periodic case. In the latter case one has [restricting ourselves to $SU(2)$ gauge theory] 2^3 degenerate electric sectors. They are related by the action of Polyakov loops winding around the volume in the space directions. But the adoption of a twist reduces this degeneracy. This has a simple physical origin. In the absence of twist, the vacua of the eight electric sectors are string states with zero momentum, but when a magnetic flux is present the string states will pick up a three-momentum equal to π/L times the Poynting vector $\boldsymbol{e}\times\boldsymbol{m}$ (modulo 2). For example, in the case $\boldsymbol{m}=(1,1,1)$ which we consider below, only two string states keep zero-momentum: $\boldsymbol{e}=(0,0,0)$ and $\boldsymbol{e}=(1,1,1)$. In this note we will only consider perturbation theory in the Fock space of one of these ground states and neglect tunneling effects between them.

We now briefly review how we compute the low-lying levels in perturbation theory, in a pure $SU(2)$ gauge theory with a twist $\boldsymbol{m}=(1,1,1)$ [5]. This twist is incorporated by the following boundary conditions on the vector potentials A_μ:

$$A_\mu(x+L\boldsymbol{e}_k) = \mathrm{i}\sigma_k A_\mu(x)(\mathrm{i}\sigma_k)^\dagger \,, \quad k=1,2,3 \,, \tag{2}$$

where L is the length of the box in the space directions, $\boldsymbol{e}_k$ is the unit vector in the kth direction, and σ_k are the Pauli matrices.

The boundary conditions can easily be diagonalized in colour space. In the basis $A_\mu = \sigma_a A_\mu^a$, eq. (2) becomes

$$A_\mu^a(x+L\boldsymbol{e}_k) = (-1)^{1+\delta_{a,k}} A_\mu^a(x) \,. \tag{3}$$

The momentum spectrum of the gauge field contains half-integral multiples of $2\pi/L$, and can be summarized by writing the total momentum as $\boldsymbol{p}=\boldsymbol{p}_\mathrm{s}+\boldsymbol{p}_\mathrm{c}$ with $\boldsymbol{p}_\mathrm{s}$ the "periodic" spatial momentum

$$\boldsymbol{p}_\mathrm{s} = \frac{2\pi}{L}(n_1, n_2, n_3) \,, \quad n_i \text{ integer} \tag{4}$$

and $\boldsymbol{p}_\mathrm{c}$ one of the set of "colour momenta"

$$\boldsymbol{p}_\mathrm{c} = \frac{\pi}{L}(0,1,-1), \frac{\pi}{L}(-1,0,1), \frac{\pi}{L}(1,-1,0) \,. \tag{5}$$

As a consequence the gluon can never have zero three-momentum. Its momentum is always a half-integral multiple of $2\pi/L$ in two directions, corresponding to a particle in a periodic box of length $2L$.

For $\boldsymbol{m}=(1,1,1)$, both the magnetic flux and the momentum spectrum are cubic and parity invariant. This follows from the twist being defined modulo 2 so that any component in the flux vector can be $+1$ or -1 without changing the results. Because of this, comparison with periodic boundary conditions is straightforward.

The energy levels we wish to compute are those of excitations of physical gauge invariant operators. Let us denote the field tensor by G and the covariant derivative by D. Then the lowest dimensional operators are $\mathrm{Tr}\, GG$, $\mathrm{Tr}\, G\mathrm{D}G$, $\mathrm{Tr}\, G\mathrm{DD}G$, $\mathrm{Tr}\, GGG$, etc. Being gauge invariant they are periodic modulo L. So if their excitations are built from perturbative one-gluon states then we must consider multigluon states with total momentum equal to an integral multiple of $2\pi/L$. In particular, to extract "masses", we choose that momentum to be zero.

The diagram in fig. 1 shows the correlation for such

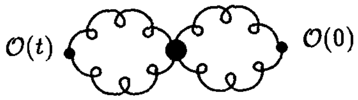

Fig. 1. Creation and annihilation of a glueball by local gauge invariant sources. The diagram shows the contribution of a two-gluon intermediate state, with a single gluon–gluon interaction. It is this four-point interaction which determines the $O(g^2)$ mass shifts.

a source. If the correlation is followed over very large times only the lowest energy states are created and absorbed by the sources, the propagation of higher ones being exponentially smaller. In between these states interact in such a way that the unphysical polarizations do not contribute at all (see section 4). We therefore discuss only matrix elements between physical helicity states.

The lowest-lying levels will be the two-gluon states with equal and opposite momenta of minimal length. Neglecting interactions, in the continuum they have energy $2\sqrt{2}\,\pi/L$. There are 24 such states: we have six momentum pairs of minimal energy as indicated in fig. 2, and each of these pairs has four helicity orientations. These decompose into irreducible representations of the cubic group (A_1, A_2, E, T_1 or T_2, of dimensions 1, 1, 2, 3 and 3 respectively) according to

$$2A_1 \oplus 2E \oplus T_1 \oplus 3T_2 \quad \text{(positive parity)},$$
$$A_1 \oplus E \oplus T_2 \quad \text{(negative parity)}. \tag{6}$$

Next come the three-gluon states with total momentum zero, and total zeroth order energy $3\sqrt{2}\,\pi/L$. There are 64 such levels which decompose as

$$4A_2 \oplus 2E \oplus 6T_1 \oplus 2T_2, \tag{7}$$

for both positive and negative parities. In both sec-

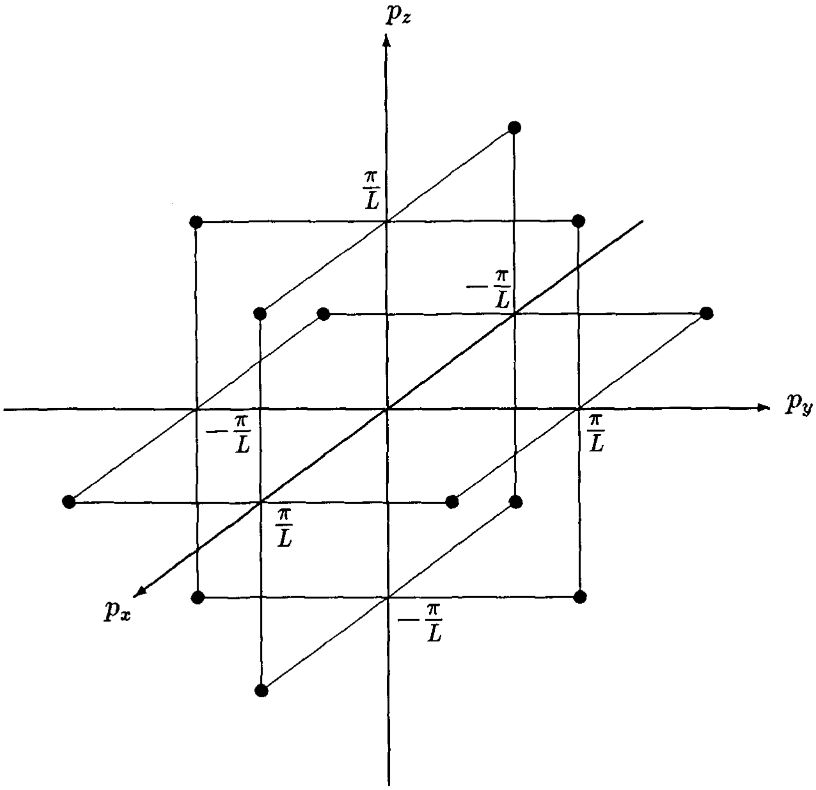

Fig. 2. The momenta of minimal length when $\boldsymbol{m}=(1,1,1)$. The lowest-lying states are constructed from one-gluon states with these momenta.

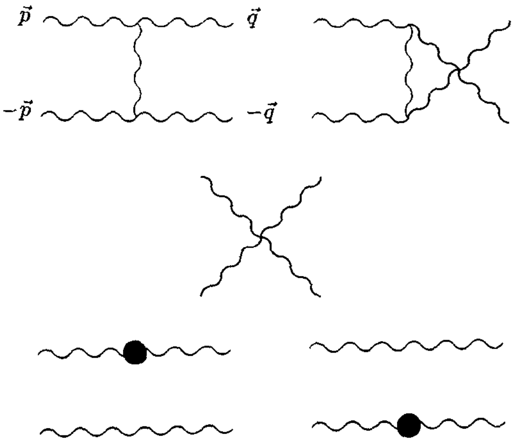

Fig. 3. Feynman diagrams required for the calculation of glueball mass shifts at $O(g^2)$. The blobs represent the one-loop self-energies.

tors, the zeroth order degeneracy is lifted by interactions.

How are these states related to the infinite volume states labeled by J^P? When restricted to the cubic group, the low spin infinite volume states decompose as

$$0 \to A_1\,, \quad 1 \to T_1\,, \quad 2 \to E \oplus T_2\,,$$
$$3 \to A_2 \oplus T_1 \oplus T_2\,. \qquad (8)$$

On the grounds of maximum symmetry of the wave function one would expect to have a light two-gluon A_1^+ in a finite volume, and a light 0^+ in the infinite volume limit. Although the infinite volume states will have an overlap with many multigluon perturbative states, we may associate these two states. Proceeding in this way, adjacent E and T_2 levels may be related to spin-2 particles, etc. Notice that it is impossible to group all the two-gluon levels into infinite volume states. At best, one T_2^+ is left over. One possibility would be for a T_2^+ and a T_1^+ from the two-gluon sector to combine with a three-gluon A_2^+ (one of which we expect to be the lightest in this sector) to form a 3^+. This is the most economical way of constructing a 3^+ state.

To $O(g^2)$ in perturbation theory, we have computed the energy shifts of the two-gluon levels with all lattice artifacts included. The results are discussed in the next section.

Apart from these "glueball masses", we can also compute the energies $E_{(100)}$ and $E_{(110)}$ of the Polyakov loops in the corresponding directions. In perturbation theory these are non-zero even at zeroth order. The simplest way to see this is to note that these loops have half-integral momenta due to the twist (they pick up a Poynting momentum $(\pi/L)\boldsymbol{e}\times\boldsymbol{m}$). In contrast, the loop in the (111) direction has integral momentum and receives only instanton contributions [6].

3. Results

In perturbation theory at $O(g^0)$ the energy of the lowest-lying two-gluon states is 2ω, where

$$\omega = \sqrt{2}\,\pi/L \quad \text{(continuum)}\,,$$
$$\omega = (2/a)\,\mathrm{arcsinh}\left(\sqrt{2}\,\sin\frac{\pi a}{2L}\right) \quad \text{(lattice)} \qquad (9)$$

is the minimal free one-particle energy (a is the lattice spacing). At $O(g^2)$, the energy shifts (the shifts of the "glueball masses") are given by the five diagrams of fig. 3. Their continuum values have been discussed in ref. [5], together with the group theoretical analysis of the spectrum. In this paper we discuss the effect of finite lattice spacing, using the conventional Wilson single plaquette action on a lattice with geometry $l^3\times\infty$. The continuum limit corresponds to $l\to\infty$, $a\to 0$, with the physical length of the box, $L=al$, held constant. Fortunately the group theoretical considerations of ref. [5] remain valid because the lattice regularization preserves cubic symmetry.

The results that we obtain are shown in fig. 4, table 1 and table 2. Levels are labeled by their cubic group representation and by their parity. The effect of finite lattice spacing a is only appreciable for the A_1^- and E^- states and $l\leqslant 8$, and is largely attributable to the tree diagrams. The lattice corrections from these diagrams die out as l^{-2}. The self-energy diagrams typically contribute 2–6 in the units of fig. 4, but they approach their continuum value slowly, as $l^{-2}\log l$.

The energies of Polyakov loops are determined by the self-energy of the gluon polarized in the direction of the loop, according to

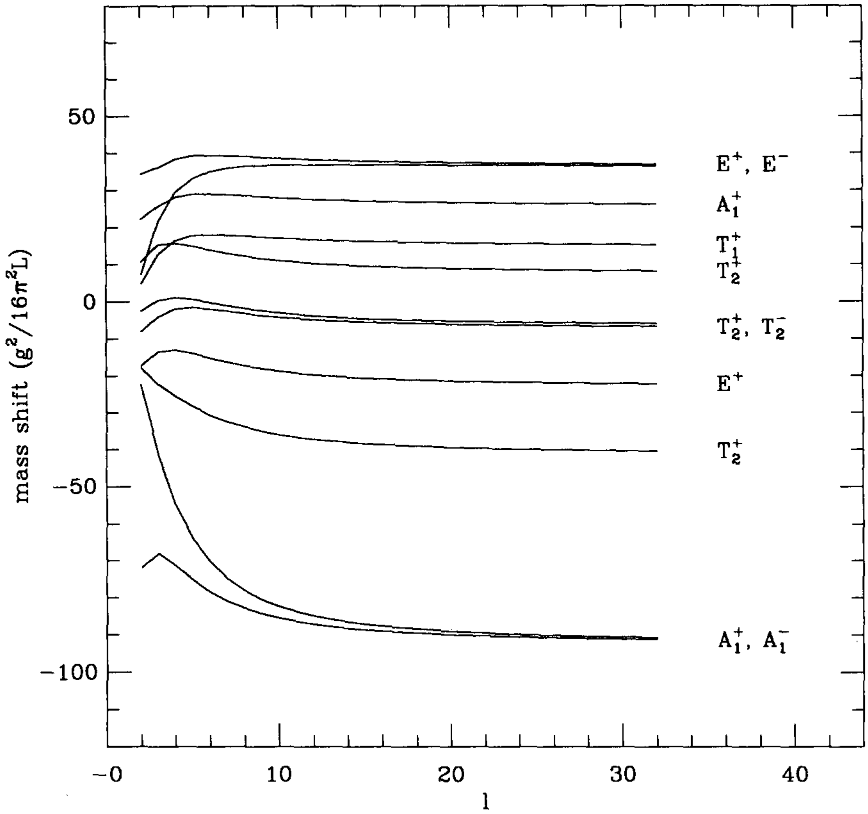

Fig. 4. The $O(g^2)$ energy shifts of the two-gluon states as a function of l.

Table 1
$O(g^2)$ shifts of the two-gluon energies ("glueball masses") in the positive parity sector. The units are $g^2/16\pi^2 L$, $L=al$.

l	A_1^+	A_1^+	E^+	E^+	T_1^+	T_2^+	T_2^+	T_2^+
2	−71.87	22.23	−17.32	34.30	4.79	−17.87	−8.08	10.71
4	−71.39	27.98	−13.08	38.26	16.39	−25.49	−1.94	15.68
8	−82.89	28.44	−17.33	39.01	17.60	−34.00	−3.21	12.23
12	−87.21	27.45	−19.73	38.15	16.63	−37.31	−4.87	10.27
16	−89.07	26.85	−20.87	37.61	16.04	−38.78	−5.73	9.31
24	−90.59	26.28	−21.85	37.07	15.46	−40.01	−6.51	8.47
32	−91.18	26.02	−22.25	36.82	15.20	−40.50	−6.84	8.12
64	−91.82	25.71	−22.70	36.53	14.89	−41.03	−7.21	7.72
∞	−92.08	25.56	−22.90	36.38	14.74	−41.26	−7.39	7.53

$$E_{(100)} = \omega - \mathcal{N}^{-1}\Sigma_{(100)}\,,$$

$$E_{(110)} = \omega - \mathcal{N}^{-1}\Sigma_{(110)}\,, \qquad (10)$$

where ω is the minimal free one-particle energy, eq. (9), $\mathcal{N}$ is the one-particle normalization

$$\mathcal{N} = 2\omega \quad \text{(continuum)}\,,$$

$$\mathcal{N} = (4/a)\sinh(\tfrac{1}{2}a\omega)\sqrt{1+\sinh^2(\tfrac{1}{2}a\omega)} \quad \text{(lattice)}\,, \qquad (11)$$

and $\Sigma_{(100)}$ and $\Sigma_{(110)}$ are the self-energy eigenvalues

Table 2
$O(g^2)$ shifts of the two-gluon energies ("glueball masses") in the negative parity sector. The units are $g^2/16\pi^2L$, $L=al$.

l	A_1^-	E^-	T_2^-
2	−22.23	7.26	−2.61
4	−54.95	29.22	1.16
8	−78.11	36.28	−1.85
12	−84.93	36.73	−3.82
16	−87.71	36.67	−4.79
24	−89.89	36.48	−5.65
32	−90.72	36.36	−6.00
64	−91.59	36.18	−6.41
∞	−91.93	36.07	−6.59

Table 3
Eigenvalues of the on-shell gluon self-energy at $p=(\pi/L)\times(0,1,-1)$ in units of g^2/L^2, labeled by their eigenvectors [$\hat{p}_\mu=2a^{-1}\sin(\frac{1}{2}ap_\mu)$, $\hat{q}$ is its parity conjugate]. The final column may also be found from the right-hand side of the Ward identity for the self-energy.

l	$\Sigma_{(100)}$	$\Sigma_{(011)}$	$\Sigma_{\hat{p}}, \Sigma_{\hat{q}}$
2	0.4923	−0.1886	−0.1515
4	0.1527	−0.2414	−0.1009
8	0.1894	−0.0756	−0.0408
12	0.2373	−0.0131	−0.0219
16	0.2609	0.0151	−0.0138
24	0.2819	0.0392	−0.0070
32	0.2907	0.0491	−0.0043
64	0.3008	0.0603	−0.0013
∞	0.3057	0.0653	0

with the polarizations as eigenvectors. The values of these eigenvalues for a variety of values of l are given in table 3.

4. Ward identities

An important feature of gauge theories is that physical states couple only to the transverse parts of gauge fields; the longitudinal modes decouple. This is the content of the Ward identities. Consequently, energies (for example) are determined by matrix elements between physical helicity states only.

We have checked that the Ward identities for *finite* lattice spacing are obeyed. In particular, the $O(g^2)$ tree diagrams should satisfy transversity (i.e. their sum should vanish if one or more of the physical polarizations are replaced by either of the unphysical polarizations, as we have explicitly verified). In view of the complicated lattice Feynman rules (especially the four gluon vertex), this is a very reassuring check on our calculation.

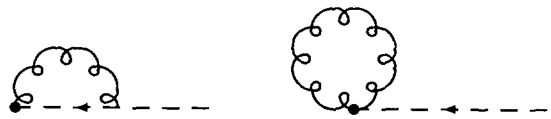

Fig. 5. The Feynman diagrams contributing to the right-hand side of the Ward identity for the self-energy.

A similar Ward identity exists for the self-energy $\Sigma_{\mu\nu}$, but on the lattice with a twist it takes a rather unexpected form. The BRST identity at one loop is

$$\hat{p}_\mu \Sigma_{\mu\nu} = I_\mu(\hat{p}_\mu \hat{p}_\nu - \delta_{\mu\nu}\hat{p}^2) , \tag{12}$$

where $\hat{p}_\mu = 2a^{-1}\sin(\frac{1}{2}ap_\mu)$, $\Sigma_{\mu\nu}$ is the sum of the self-energy diagrams, and I_μ is given by the two diagrams of fig. 5.

In the continuum limit ($a\to 0$, $\hat{p}_\mu \to p_\mu$), I_μ is proportional to p_μ and therefore the right-hand side of eq. (12) vanishes. On the lattice with a twist this is *not* the case. Nevertheless eq. (12) in this lattice form still forbids the propagation of unphysical polarizations. We computed both sides of the identity on-shell ($\hat{p}^2=0$) on lattices up to size $64^3\times\infty$, and found excellent agreement. The third column of table 3 gives the calculated values.

5. Inclusion of fermions

Fermions do not fit comfortably into a gauge theory with twisted boundary conditions. Typically we want fermions in the fundamental representation of the gauge group, but twisted boundary conditions can only be applied to the adjoint representation, which is insensitive to the group centre. In order to overcome this problem, we introduce two flavours of fermion [N flavours in an SU(N) theory] and include a flavour transformation in the boundary conditions:

$$\psi(x+e_kL) = (i\sigma_k)\otimes(i\sigma_k)^\dagger\psi(x) ,$$
$$\bar{\psi}(x+e_kL) = \bar{\psi}(x)(i\sigma_k)^\dagger\otimes(i\sigma_k) , \tag{13}$$

where the first matrix acts on colour, the second on flavour. The flavour transformation removes the sign ambiguity induced by the colour transformation.

At $O(g^2)$ fermions only enter our calculation through the gluon self-energy. We have computed the effect of the fermions in the continuum limit and find that their contribution increases the self-energy by precisely one quarter. This shifts the energy levels down a little, but has very little effect on their relative separation.

6. Conclusions

We have studied the finite lattice spacing corrections to the lowest-lying energy levels of SU(2) lattice gauge theory with twist $\boldsymbol{m}=(1,1,1)$, calculated to $O(g^2)$ in perturbation theory. In general, $O(a^2)$ effects are small even for $l=4$. This is not true in the negative parity sector, however. In particular the perturbative shifts of the A_1^- and E^- states suffer substantial lattice corrections at $l=4$. At small l (a large relative to L) the dominant corrections come from tree diagrams, but these effects die out as l^{-2}. Quantum corrections from the self-energy, which are smaller, approach their continuum values only logarithmically (as $l^{-2}\log l$) and so lattice effects here persist to large l.

One of the motivations for this study was to reconcile our earlier perturbative results in the continuum with the Monte Carlo data of ref. [7]. Meanwhile, these simulations have progressed and their is now data at various values of $\beta=4/g^2$ in the range 2.5–4.7 [8]. For the positive parity sector, there is qualitative agreement on the spectrum in the sense that the perturbative results fit the Monte Carlo data if we make a rescaling of the coupling constant. However, it is only at $\beta=4.7$, the weakest coupling yet examined, that simulation and perturbation theory are quantitatively in accord within errors, for both positive and negative parity sectors (except for the A_1^+), and without any ad hoc rescaling of the coupling. For purely periodic boundary conditions [9] on the other hand, tunneling-improved perturbation theory works well from about $\beta=2.5$, in small volumes.

We have also studied the perturbative corrections to the energies of Polyakov loops. For the values of β at which the glueball spectrum indicates the validity of perturbation theory, these corrections are very small, so we expect $E_{(100)}:E_{(110)}:E_{(111)}=1:1:0$. As the physical size L grows, we expect a transition to the string regime $E_{(100)}:E_{(110)}:E_{(111)}=1:\sqrt{2}:\sqrt{3}$. These expectations are confirmed by simulations on lattices of spatial size 4^3, 8^3 and 12^3 [8], which we hope will be extended in the future.

A second, and more important motivation for our work in refs. [3,5], which we continue here, is the intuitive expectation that finite size effects should be smaller with twisted boundary conditions. This idea is based on the absence of zero-momentum modes, and the fact that the one-particle momenta are closer together, making the box seem bigger. Our perturbative results tend to confirm this, in the sense that the spectrum does not change qualitatively from its perturbative form as we go to larger and larger volumes. In particular, the 0^+ ($\sim A_1^+$) starts out and remains the lightest state, followed, in the positive parity sector, by the 2^+ ($\sim E^+ \oplus T_2^+$). This is in sharp contrast to the position with periodic boundary conditions. However, in the negative parity sector the 2^- ($\sim E^- \oplus T_2^-$) behaves in a qualitatively similar manner to its counterpart in the periodic case. Also the 0^- ($\sim A_1^-$) suffers comparable finite size effects.

Finally we make a comment about the mass gap m. In an idealized lattice simulation, masses are extracted in terms of the variable $z=mL$. The continuum limit is extracted by simultaneously increasing l and decreasing the bare coupling g^2, while keeping a fixed value of z. Eventually the infinite volume limit is reached by increasing z. However, our lowest order perturbative result (see first column of table 1) indicates that z is smaller than 2ω, where ω is the minimal one-particle energy, eq. (9). Therefore higher order perturbative corrections and non-perturbative effects are instrumental in achieving the physical range of z values [10]. In order to see what is going on we appeal to the data of ref. [8] and show them in fig. 6. The horizontal axis exhibits the length L of the box in terms of the inverse large string tension. For small enough values of this variable all the quantities mL, in particular the mass gap 0^+, coincide, within errors, with the predictions in tables 1 and 2. As L is decreased, the mass gap at first decreases and then, beyond a critical value ($L\sim 1/\sqrt{\kappa}$, where κ is the large volume string tension), starts to increase. Thus the very small L "coulombic" behaviour of the gap, that would have rendered it unstable at large L,

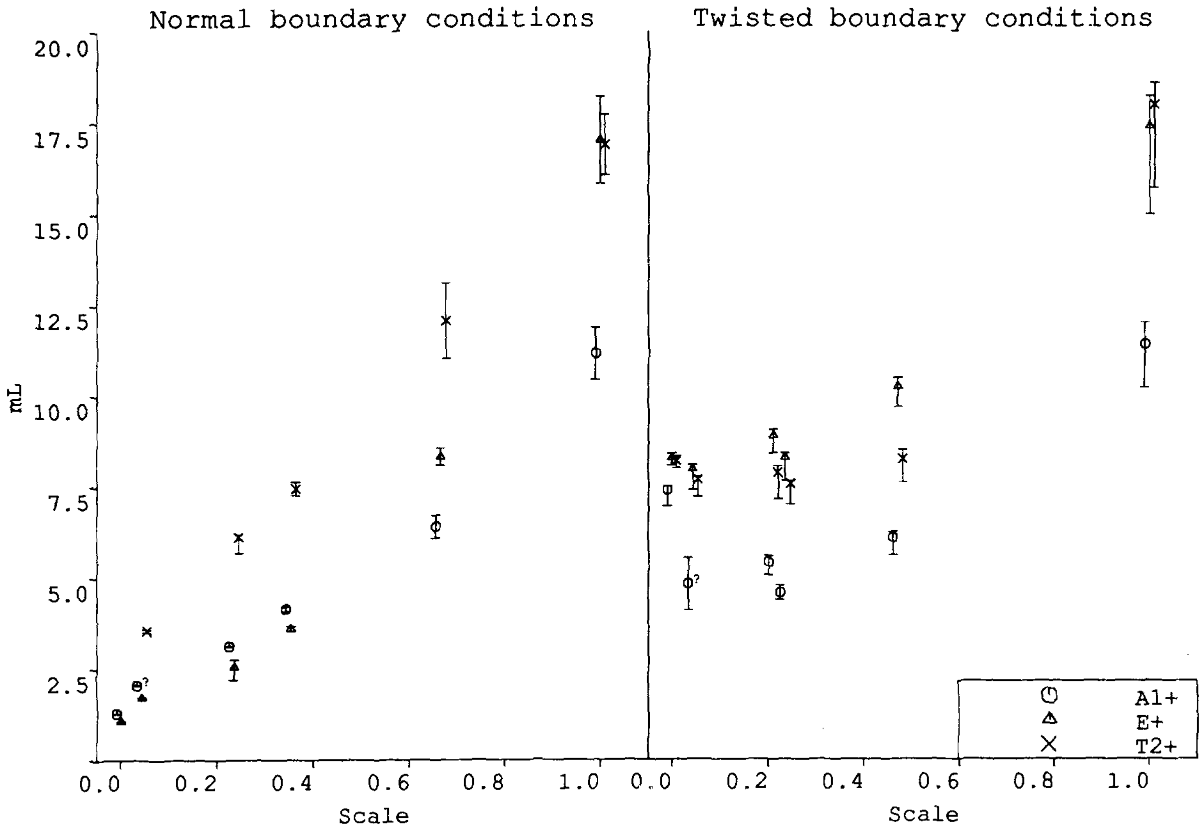

Fig. 6. Monte Carlo data for the glueball spectrum with both periodic and twisted boundary conditions. The horizontal scale is the box length in units of $2.98/\sqrt{\kappa}$, where κ is the large volume string tension at the same β. Taken from ref. [8].

is overcome at this point. To understand how is the great challenge.

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References

[1] B. Berg and A. Billoire, Phys. Rev. D 40 (1989) 550.
[2] M. Lüscher, in: Progress in field theory, eds. G. 't Hooft et al. (Plenum, New York, 1984); Commun. Math. Phys. 104 (1986) 177.
[3] A. González-Arroyo and C.P. Korthals Altes, Nucl. Phys. B 311 (1988) 433.
[4] G. 't Hooft, Nucl. Phys. B 153 (1979) 141.
[5] D. Daniel, A. González-Arroyo, C.P. Korthals Altes and B. Söderberg, Phys. Lett. B 221 (1989) 136.
[6] M. García Pérez, A. González-Arroyo and B. Söderberg, Phys. Lett. B 235 (1990) 117.
[7] P.W. Stephenson and M. Teper, Nucl. Phys. B 327 (1989) 307.
[8] P. Stephenson, private communication; and to be published.
[9] P. van Baal, Phys. Lett. B 224 (1989) 397.
[10] For an alternative view see T.H. Hanson, K. Johnson and C. Peterson, Phys. Rev. D 26 (1989) 2069.